JOURNAL OF ALGEBRA 169, 587–604 (1994)

Finitely Generated P.I. Rings of Global Dimension Two

AMIRAM BRAUN*

Department of Mathematics, Haifa University, Haifa 31905, Israel

AND

CHARUDATTA R. HAJARNAVIS

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

Communicated by J. T. Stafford

Received January 7, 1992

1. INTRODUCTION

In this paper we mainly study (semi)prime Noetherian P.I. (polynomial identity) rings of global dimension 2. We have the following.

THEOREM A. *Let* $R = A\{x_1, \ldots, x_k\}$ *be a semi-prime Noetherian finitely generated P.I. ring, where A is a subring of* $Z(R)$, *the center of R. Suppose that* gl.dim $R \leq 2$. *Then R is a finite module over its Noetherian center.*

Note that we do not assume A to be Noetherian. This theorem may be viewed as a generalization of the result obtained by Robson and Small [RS1], where they showed that an hereditary prime Noetherian P.I. ring is a finite module over its center and the center is a Dedekind domain. However, any attempt to remove the finite generation assumption on R, in Theorem A, is doomed to failure. In fact, we have the following.

EXAMPLE B. There exists a prime Noetherian P.I. ring R, with gl.dim $R = 2$, such that R is *not* integral over $Z(R)$.

* This work was partially supported by a S.E.R.C. grant at the University of Warwick. The first author thanks the Mathematics Institute at the University of Warwick for its hospitality.

0021-8693/94 $6.00

Copyright © 1994 by Academic Press, Inc.
All rights of reproduction in any form reserved.

481/169/2-17

Hence, Theorem A seems to be the best possible in this direction.

Our next result seems to be an interesting corollary of Theorem A (we refer the reader to [BGL] or [DR] for the terminology).

THEOREM C. *Let Λ be a finite-dimensional tame hereditary k-algebra (k being a field). Let $\pi(\Lambda)$ be the preprojective algebra of Λ. Then $\pi(\Lambda)$ is a finite module over its Noetherian center.*

Proof. By [BGL], $\pi(\Lambda)$ is a prime Noetherian affine P.I. k-algebra and gl.dim $\pi(\Lambda) = 2$. We now apply Theorem A. Q.E.D.

Our research was partly motivated by the following question of Vasconcelos [V, p. 21]:

Let R be a prime Noetherian ring which is a finite module over its center $Z(R)$. Suppose that gl.dim R = K.dim $R = 2$. Is $Z(R)$ an integrally closed ring?

The following result answers the question negatively.

EXAMPLE D. There exists, over any field, a prime Noetherian affine P.I. ring having the following properties:

(i) gl.dim R = K.dim $R = 2$, p.i.deg$(R) = 4$.

(ii) $Z(R)$ is not integrally closed.

(iii) $Z(R)$ is not Cohen–Macaulay.

While Example D is somewhat disappointing, there are many special cases when $Z(R)$ is a Krull domain (and therefore $Z(R)$ is Cohen–Macaulay). A similar example, with p.i.deg $(R) = 3$, can be constructed (the field in this example however is not arbitrary).

The situation of Vasconcelos's question, in the case of p.i.deg$(R) = 2$, is more interesting and positive results can be obtained. More generally, we have the following.

THEOREM E. *Let R be an affine Noetherian prime P.I. ring. Suppose that*

(i) gl.dim R = K.dim $R = 2$.

(ii) p.i.deg$(R/M) \geq \frac{1}{2}$ p.i.deg(R), *for every maximal ideal M in R.*

Then $Z(R)$ is a Krull domain and R is a finite $Z(R)$-module.

Recall that by [GS], K.dim $R \leq$ gl.dim R whenever R is a Noetherian P.I. ring. We also find (Example 20) a prime Noetherian affine P.I. ring R with gl.dim R = p.i.deg$(R) = 2$ and K.dim $R = 1$, such that $Z(R)$ is not integrally closed. This shows that the assumption of equality in Theorem E(i) is necessary. The affine assumption, in Theorem E, is also shown to be necessary (Example 21).

Finally, nothing similar is valid in higher dimensional rings. In fact, we have the following.

EXAMPLE F. There exists a prime Noetherian affine P.I. ring R, with K.dim R = gl.dim $R = 3$, such that R is *not* a finite $Z(R)$-module.

The paper is organized as follows. After an introduction and preliminaries in Sections 1 and 2, we prove in Section 3 our main result, namely Theorem A, in the prime case. In Section 4 we describe Example D, which answers, negatively, Vasconcelo's question. In Section 5 we construct a prime Noetherian P.I. ring R with gl.dim $R = 2$ such that R is not integral over $Z(R)$. Section 6 provides a proof of Theorem E and some examples. In Section 7 we describe Example F.

2. Preliminaries and Notation

R denotes an associative not necessarily commutative ring with identity. We follow the standard usage and terminology of ring theory and refer the reader to [MR] for the background material. Some of the notation used is explained below.

$A \supset B$ means that B is properly contained in A.

$\mathscr{C}(I)$ is the set of all elements of R which are regular modulo the ideal I.

$\Pi(p)$ stands for the idealiser of the right ideal p.

l.gl.dim R is the left global dimension of R. When the left and right global dimensions coincide, we denote the common value by gl.dim R.

K.dim R stands for the classical Krull dimension of R.

pr.dim M denotes the projective dimension of the module M.

$Q(R)$ is the full quotient ring of R.

$t(R)$ is the number of distinct idempotent ideals in R.

$Z(R)$ denotes the centre of R.

p.i.deg(R) is the smallest integer n such that the semi-prime ring R satisfies all the identities of $n \times n$ matrices.

Let R be a prime right Noetherian ring with a right quotient ring $Q(R)$. Let I be a non-zero ideal of R. We define $I^* = \{q \in Q(R) \mid qI \subseteq R\}$. Note that I^* is a ring when $I^*I = I$. Likewise $I^+ = \{q \in Q(R) \mid Iq \subseteq R\}$. It is well-known that by the dual basis lemma I_R (respectively, $_RI$) is projective if and only if $1 \in II^*$ (respectively, $1 \in I^+I$).

We make frequent use of the following (see, e.g., [K, Theorem 2, p. 169]). Let $0 \to A \to B \to C \to 0$ be an exact sequence of right (left) R

modules. If $\text{pr.dim}_R A < \text{pr.dim}_R B$, then $\text{pr.dim}_R B = \text{pr.dim}_R C$. If $\text{pr.dim}_R A > \text{pr.dim}_R B$, then $\text{pr.dim}_R C = 1 + \text{pr.dim}_R A$.

Where relevant, the absence of the adjectives right or left implies two-sided assumptions. We also frequently employ Auslander's theorem that in a Noetherian ring the left and right global dimensions coincide and Cauchon's theorem that in a prime P.I. ring left Noetherian and right Noetherian are equivalent conditions. Finally, the term affine ring stands for a finitely generated ring over a central subfield.

3. The Proof of Theorem A—The Prime Case

LEMMA 1. *Let A be a prime ring and $q = c^{-1}d$, $c, d, \in Z(R)$. Set $I = \{x \in R \mid qx \in R\}$. Then $I \cong cR \cap dR$ as a left and right R-module.*

Proof. Clearly I is a two-sided ideal in R. The map $\varphi: I \to cR \cap dR$, given by $\varphi(x) = dx$, affords the desired isomorphism. Q.E.D.

COROLLARY 2. *Let R be a prime P.I. ring with* $\text{gl.dim}\, R = 2$. *Let c, d, q, I be as in the previous lemma. Then I_R, ${}_RI$ are projective.*

Proof. By the previous lemma, it suffices to show that $cR \cap dR$ is projective as left and right R-modules.

We have the following exact sequence

$$0 \to cR \cap dR \to cR \oplus dR \to cR + dR \to 0.$$
$$(x, y) \to x + y.$$

Now, if $cR \cap dR$ is not projective, then $\text{pr.dim}_R\, cR + dR \geq 2$, which implies that $\text{pr.dim}_R\, (R/(cR + dR)) \geq 3$, a contradiction to $\text{gl.dim}\, R = 2$. Q.E.D.

LEMMA 3. *Let R be a prime Noetherian P.I. ring and X a clique of height one primes. Then X is finite and hence localizable.*

Proof. Let $P \in X$ and $p \equiv P \cap Z(R)$. Then, for every member Q of X, $Q \cap Z(R) = p$ and all members of X are minimal over pR. Hence X is finite, and by [J. Chap. 7] X is localizable. Q.E.D.

Notation. Given a clique X, by R_X we denote the ring obtained from R by localizing at $\mathcal{C}(X)$ (when localization is possible).

THEOREM 4. *Let R be a prime Noetherian P.I. with* $\text{gl.dim}\, R = 2$. *Suppose that for every right simple R-module T we have either*

(i) $\text{pr.dim}\, T_R = 2$, or

(ii) $\text{pr.dim}\, T_R = 1$, and $M^*M = R$ where $M = r\text{-ann}_R T$.

Then R is integral over $Z(R)$ and $Z(R)$ is a Krull domain.

Proof. We first remark that if $M^*M = R$ and M is maximal (this being a consequence of (ii)), then by [H] height $(M) = 1$, M is localizable, and $M^+ = M^*$. Consequently R_M, the localization of R at $\mathscr{C}(M)$, is hereditary. Therefore, by [RS1, Theorem 2] (or [MR, Theorem 13.9.16]), $Z(R_M)$ is a Dedekind domain.

Second, let P be a height-one prime in R which is not maximal. Then by [J, Theorem 5.2.9] each member of X, the clique of P, is not maximal. Now, since $\text{K.dim } R \leq \text{gl.dim } R = 2$ [GS, Theorem D], we have that each member of X is of height one. Consequently, by the previous lemma, X is finite and localizable. Therefore, using [BW, Corollary 5, as in the proof of Theorem 8] and the fact that X consists of non-maximals, we get that $\text{gl.dim } R_X \leq 1$. That is, R_X is hereditary and therefore $Z(R_X)$ is a Dedekind domain by [RS1]. We claim that

$$Z(R) = \bigcap_X Z(R_X) \bigcap_Y Z(R_Y), \tag{*}$$

where X runs through clique (P), height $(P) = 1$, P is a non-maximal prime ideal in R, and Y runs through clique (M), M maximal in R, $M^*M = R$.

We next show how (*) implies the required result. Indeed, by the previous discussion, for each X, Y we have that $Z(R_X)$ and $Z(R_Y)$ are Dedekind domains and hence completely integrally closed. Consequently, their intersection $Z(R)$ is completely integrally closed and, by [Sch], R is integral over $Z(R)$. However, by [B2], this implies that $Z(R)$ is a Krull domain.

We now establish (*). Let $q \in \bigcap_X Z(R_X) \bigcap_Y Z(R_Y)$ and $q \notin Z(R)$. Clearly $q \in Z(Q(R)) = Q(Z(R))$. Then $q = c^{-1}d$, c, $d \in Z(R)$. This instantly implies that $q \notin R$. Let $I = \{r \in R \mid qr \in R\}$. Hence, I is a proper two sided ideal in R and, by Corollary 2, I_R is projective.

Let P be a prime ideal in R such that $P \supseteq I$. If P is not maximal, then height $(P) = 1$, hence $X \equiv$ clique (P) appears in the intersection (*). Therefore $q \in Z(R_X)$. Equivalently, there exists $s \in \mathscr{C}(X)$ satisfying $sq \in R$. Observe that $s \in \mathscr{C}(P)$, since $\mathscr{C}(X) \subseteq \mathscr{C}(P)$. However, $s \in I \subseteq P$ which contradicts $s \in \mathscr{C}(P)$. We therefore have that R/I is Artinian. Let $T \subseteq R/I$ be a simple right R-module. Since $\text{pr.dim}_R\, (R/I) = 1$, $\text{gl.dim } R = 2$, the exact sequence,

$$0 \to T \to \frac{R}{I} \to \frac{R/I}{T} \to 0,$$

implies that $\text{pr.dim } T_R = 1$ (it cannot be 0). Let $M \equiv r\text{-ann}_R T$. We have $R/M \cong_R T^{(n)}$, for some n. Hence, M_R is projective, which implies by (ii) that $M^*M = R$ and $M \in Y$. Now, $I = r\text{-ann}_R\, (R/I) \subseteq r\text{-ann}_R\, T = M$. Moreover, $\{M\} =$ clique (M) and $q \in R_M$ which provides an element s, $s \in \mathscr{C}(M)$ and satisfying $sq \in R$. That is, $s \in I \subset M$, which contradicts

$s \in \mathscr{C}(M)$. All in all, the right-hand side of (*) is contained in $Z(R)$. The other inclusion in (*) is trivial. Q.E.D.

LEMMA 5. *Let R be a semi-prime left Noetherian ring and M an ideal in R containing a regular element. Then M^* is a finitely generated left R-module.*

Proof. For any regular element m in M, we have $M^* \cong M^*m \subseteq R$. Q.E.D.

LEMMA 6. *Let R be a prime P.I. ring, M a maximal ideal in R, and $M \cap Z(R) \equiv q$. Suppose that*

(1) $M^*M = M$.

(2) *M^* is integral over $Z(M^*)$.*

Then $v \cap Z(R) = q$, for every prime ideal v in $Z(M^*)$ satisfying $v \supseteq q$.

Proof. Recall that q is an ideal in $Z(M^*)$ and M is a two-sided ideal in $RZ(M^*)$. Now, by (2), $RZ(M^*)$ is integral over $Z(M^*)$ and therefore $RZ(M^*)/M \supset Z(M^*)/q$ is an integral extension. Set $\bar{v} = v/q$. Then $\bar{v}$ is a prime ideal in $Z(M^*)/q$. Hence, by G.U., there exists a prime ideal $\bar{V}$ in $RZ(M^*)/M$, satisfying $\bar{V} \cap Z(M^*)/q = \bar{v}$. That is, there exists a prime ideal V in $RZ(M^*)$, $V \supseteq M$, $\bar{V} = V/M$, satisfying $V \cap Z(M^*) = v$. However, $V \cap R$ is an ideal in R and $V \cap R \supseteq M$ implies, since M is maximal, that $V \cap R = M$. Finally, $v \cap Z(R) = (V \cap Z(M^*)) \cap Z(R) = V \cap Z(R) = (V \cap R) \cap Z(R) = M \cap Z(R) \equiv q$. Q.E.D.

PROPOSITION 7. *Let R be a prime P.I. ring, M a maximal ideal in R, and $q \equiv M \cap Z(R)$. Suppose that*

(1) $M^*M = M$,

(2) *$Z(M^*)$ is Noetherian.*

Then

(i) *q satisfies the AR property in $Z(R)$, and*

(ii) *$\bigcap_i (v_p + q_p^i) = v_p$, for all primes p, v in $Z(R)$ with $p \supseteq q$ and $v \subseteq p$.*

Proof. (i) Let I be an ideal in $Z(R)$. $Z(M^*)$ is Noetherian, hence q, being an ideal in $Z(M^*)$, satisfies the AR property in $Z(M^*)$. Consequently, for some n, $q^n \cap IZ(M^*) \subseteq qIZ(M^*) = IqZ(M^*) = Iq = qI$. Hence, $q^n \cap I \subseteq q \cap IZ(M^*) \subseteq qI$.

(ii) Nothing is changed if we localize R, M^*, M with respect to $Z(R) - p$. Consequently q_p satisfies the AR property in $Z(R)_p = Z(R_p)$. It is now standard, using the AR property, that $\bigcap_i (v_p + q_p^i) = \{x \in Z(R)_p \mid x(1 - a) \in v_p, \text{ some } a \in q_p\}$. However, $Z(R)_p$ is local with maximal p_p and

$q_p \subseteq p_p$, which implies that $1 - a$ is invertible in $Z(R_p)$, that is, $x \in v_p$. Consequently, $\bigcap_i (v_p + q_p^i) = v_p$. Q.E.D.

PROPOSITION 8. *Let R be a prime Noetherian P.I. ring, M a maximal ideal in R and $q \equiv M \cap Z(R)$. Suppose that*

(1) $M^*M = M$,

(2) *$Z(M^*)$ is Noetherian.*

Then q is a maximal ideal in $Z(R)$.

Proof. Suppose that q is not maximal and let p be a prime ideal in $Z(R)$ with $p \supset q$. If $pZ(M^*) \neq Z(M^*)$, we pick v, a prime ideal in $Z(M^*)$ such that v contains $pZ(M^*)$. Hence, by Lemma 6 we get $p \subseteq v \cap Z(R) = q$, a contradiction. Consequently $pZ(M^*) = Z(M^*)$.

Observe that by Lemma 5 the inclusion $R \subseteq RZ(M^*) \subseteq M^*$ implies that $RZ(M^*)$ is a finite central extension of R. Now by (2), M^* is finite over $Z(M^*)$. Also, it is easily verified that $Z(RZ(M^*)) = Z(M^*)$, which implies that $RZ(M^*)$ is a finite module over $Z(M^*)$.

If $pR \neq R$, then there exists a prime ideal P in R satisfying $P \supseteq pR$ and hence a prime ideal P' in $RZ(M^*)$ (this being a finite central extension of R), such that $P' \cap R = P$. Hence $p' \equiv P' \cap Z(RZ(M^*)) = P' \cap Z(M^*)$ satisfies $p' \supseteq pZ(M^*) = Z(M^*)$, a contradiction. We therefore have $pR = R$ (and that the same holds for every prime p_1 in $Z(R)$, $p_1 \supset q$).

If, for every prime v in $Z(R)$ with $v \subset p$, we have $v \supseteq q$, then we can pick $x \in p$, $x \notin q$, and a prime ideal $p_0 \subseteq p$, where p_0 is minimal with respect to containing $xZ(R) \neq Z(R)$. Consequently, ${p_0}_{p_0}$ is minimal over $xZ(R)_{p_0}$. However, $xR_{p_0} \cap Z(R_{p_0}) = xZ(R_{p_0})$, and $Z(R_{p_0}) = Z(R)_{p_0}$ implies that ${p_0}_{p_0}$ is contracted from a prime ideal P_{p_0} in R_{p_0}. That is, $P \cap Z(R) = p_0$, and therefore $p_0R \neq R$, a contradiction by the previous paragraph since $p_0 \supset q$.

Consequently, there exists a prime ideal v in $Z(R)$, satisfying $v \subset p$, $v \nsubseteq q$, and $q \nsubseteq v$. Now $pZ(M^*) = Z(M^*)$ implies the existence of $g_1, \ldots, g_t$, elements in p, such that $\sum_1^t g_ir_i = 1$, where $r_i \in Z(M^*)$ for $i = 1, \ldots, t$. Let p_0 be a prime in $Z(R)$, minimal with respect to containing $q + v$, and $p_0 \subseteq p$ (we can find this by Zorn's lemma). So, without loss of generality, we may assume that p is minimal over $v + q$. Consequently, p_p is a nil ideal modulo $v_p + q_p$. Hence, there exists a k so that $g_i^k \in v_p + q_p$, for $i = 1, \ldots, t$. Let $w = g_1Z(R)_p + \cdots + g_tZ(R)_p$. Then $w^{kt} \subseteq v_p + q_p$ and therefore $w^{kti} \subseteq (v_p + q_p)^i \subseteq v_p + q_p^i$ for all i. That is, $\bigcap_i w^{kti} \subseteq \bigcap_i (v_p + q_p^i) = v_p$, where the last equality is true by Proposition 7.

Finally, $\sum_{i=1}^t g_ir_i = 1$ implies that $wZ(M_p^*) = Z(M_p^*)$. Hence $q_p = Z(M_p^*)q_p = wZ(M_p^*)q_p = wq_p$ and, by iterations, $q_p \subseteq \bigcap_i w^i$. Consequently, by the previous paragraph, $q_p \subseteq v_p$. Therefore $q \subseteq v$, which contradicts the choice of v. Q.E.D.

PROPOSITION 9. *Let $R = A\{x_1, \ldots, x_k\}$ be a prime Noetherian P.I. ring where $A \subseteq Z(R)$. Let M be a maximal ideal in R. Suppose that*

(1) $M^*M = M$,

(2) $Z(M^*)$ *is Noetherian.*

Then $Z(R)$ is Noetherian and R is a finite $Z(R)$-module.

Proof. By Lemma 5, M^* is a finitely generated left R-module. Consequently $RZ(M^*)$ is a finite central extension of R and obviously M is a two-sided ideal in $RZ(M^*)$. Therefore, $RZ(M^*) = A\{y_1, \ldots, y_s\}$ and $RZ(M^*)/M$ is a finite central extension of R/M. We may assume, without loss of generality, that $A = Z(R)$. Now, by Proposition 8, $q \equiv M \cap Z(R)$ is maximal in $Z(R)$, so we get, by the P.I. Hilbert's Nullstellensatz (e.g., [MR, Theorem 13.10.3]), that R/M is finite dimensional over $Z(R)/q$. Therefore $RZ(M^*)/M$ is finite dimensional over $Z(R)/q$. Consequently, $Z(M^*)/q$ is finite dimensional over $Z(R)/q$ which implies that $Z(M^*)$ is integral over $Z(R)$. Consequently, M^* is integral over $Z(R)$ which implies that R is integral over $Z(R)$. Finally, Shirshov's theorem (e.g., [MR, Corollary 13.8.9]) furnishes the remaining assertions of the proposition. Q.E.D.

Recall that $t(R)$ is the number of proper idempotent ideals in R.

PROPOSITION 10 [RS2]. *Let R be a Noetherian P.I. ring. Then $t(R) < \infty$.*

Our next lemma is essentially Proposition 1.8 of [ER].

LEMMA 11. *Let M be a maximal ideal of a prime Goldie ring R, such that M_R is projective and $M^*M = M$. Then there is a one to one correspondence between idempotent ideals I in R with $I \subseteq M$ and the idempotent ideals in M^*.*

Proof. Note that since M_R is projective, $1 \in MM^*$. Therefore $M = 1M \subseteq MM^*M = M^2$, so M is an idempotent. The rest of the proof is exactly as in [ER, Proposition 1.8]. Indeed the maps are given by $\varphi: I \to IM^*$ and $\psi: J \to JM$ for every idempotent ideal J in M^*.

COROLLARY 12. *With the assumptions of Lemma 11, we have $t(M^*) < t(R)$.*

Proof. M is a proper idempotent ideal in R which is mapped by φ to $MM^* = M^*$. Consequently $t(M^*) =$ the number of proper idempotent ideals in $M^* =$ the number of idempotent ideals in R which are properly contained in $M \leq t(R) - 1$. Q.E.D.

We now can prove one of our main results.

THEOREM 13. *Let $R = A\{x_1, \ldots, x_k\}$ be a prime Noetherian finitely generated P.I. ring, where $A \subseteq Z(R)$, but A is not necessarily Noetherian. Suppose that* gl.dim $R \leq 2$. *Then R is a finite module over its Noetherian center.*

Remark. This theorem can be proved as well in the semi-prime case. However, the extra generality may not be worth the extra technicalities, and so the proof is omitted.

Proof. The proof is by induction on $t(R)$ which is finite by Proposition 10. We first assume that $t(R) = 0$. Let M be a maximal ideal in R. If M_R is projective and $M^*M = M$, then since $1 \in MM^*$ (M_R being projective) we have $M \subseteq MM^*M = M^2$. Hence $M = M^2$ which contradicts $t(R) = 0$. Consequently, for every maximal M in R we either have pr.dim $M_R = 1$, or pr.dim $M_R = 0$ and $M^*M = R$. Now Theorem 4 is applicable and we conclude that R is integral over $Z(R)$ (and $Z(R)$ is a Krull domain). By a standard argument we deduce that $Z(R) = A[c_1, \ldots, c_k]$ and $Z(R)$ is Noetherian, which implies that R is finite over $Z(R)$.

We may therefore assume that there exists a maximal ideal M in R, such that $M^*M = M$ and M_R is projective (otherwise, we can finish, as above, by Theorem 4). Now by Corollary 12, we have that $t(M^*) < t(R)$. Also $MM^* = M^*$ implies by [RS3, Theorem 5(i)] that gl.dim $M^* \leq$ gl.dim $R \leq 2$. Moreover, M^* being a finite left module over R by [J, Theorem 5.2.2] implies that M^* is Noetherian and is a finitely generated A algebra. Consequently, by induction, M^* is a finite module over its Noetherian center $Z(M^*)$. The theorem now follows instantly from Proposition 9. Q.E.D.

A simpler proof of Theorem A *in the prime affine case.* One should remark that our proof of Theorem A, even in the prime case, is quite involved. However, it simplifies considerably if one assumes that A is, in addition, a field. Thus in this case Proposition 9 becomes very easy and Lemma 6 and Propositions 7, 8 are not needed. In this case a different proof can also be given, provided one uses a certain (Schelter) integrality result which was proved in [BV].

4. $Z(R)$ IS NOT INTEGRALLY CLOSED

We construct here a ring R with the following properties.

EXAMPLE 14. (i) R is a prime Noetherian P.I. ring of P.I. degree 4, affine over an arbitrarily chosen field.

(ii) gl.dim R = K.dim $R = 2$.

(iii) $Z(R)$ is not integrally closed.

(iv) $Z(R)$ is not Cohen–Macaulay.

Let F be a field and $A = M_2(F[x, y])$ the 2×2 matrices over $F[x, y]$. We choose $a_1, a_2 \in F$, $a_1 \neq a_2$, and set $m_1 = (x - a_1, y)$, $m_2 = (x - a_2, y)$, two different maximal ideals in $F[x, y] = Z(A)$. Now $A/m_iA \cong M_2(F)$, so we choose $\rho_i \supset m_iA$, ρ_i a right maximal ideal in A. Also, $A/\rho_1 \not\cong_A A/\rho_2$ is a consequence of $m_1 = \rho_1 \cap Z(A) \neq \rho_2 \cap Z(A) = m_2$. It is trivial (by choice) that $\dim_F (A/\rho_i) = 2$ and $\mathrm{End}_A(A/\rho_i) \cong F$ for $i = 1, 2$.

We claim that $A(\rho_1 \cap \rho_2) = A$. If not then we can find a maximal ideal P in A such that $P \supseteq \cdot A(\rho_1 \cap \rho_2)$. Therefore $P \supseteq \rho_1 \cap \rho_2$. Hence $\dim_F (A/(\rho_1 \cap \rho_2)) \geq \dim_F (A/P) \geq 4$. However, $A/\rho_1 \cap \rho_2 \cong A/\rho_1 \oplus A/\rho_2$. So $\dim_F A/(\rho_1 \cap \rho_2) = 4$. Thus $P = \rho_1 \cap \rho_2$ which contradicts $(A/\rho_1) \not\cong_A (A/\rho_2)$. Let $B \equiv \mathbb{I}(\rho_1 \cap \rho_2)$ be the idealizer of $\rho_1 \cap \rho_2$. Then

$$\begin{aligned}
\frac{B}{\rho_1 \cap \rho_2} &\cong \mathrm{End}_A\left(\frac{A}{\rho_1 \cap \rho_2}\right) \cong \mathrm{End}_A\left(\frac{A}{\rho_1} \oplus \frac{A}{\rho_2}\right) \\
&\cong \mathrm{End}_A\left(\frac{A}{\rho_1}\right) \oplus \mathrm{End}_A\left(\frac{A}{\rho_2}\right), \qquad \text{since } \frac{A}{\rho_1} \not\cong_A \frac{A}{\rho_2}, \\
&\cong F \oplus F.
\end{aligned}$$

Hence $B/(\rho_1 \cap \rho_2)$ is semi-simple Artinian and so by [G, Proposition 1.1] we conclude that ${}_B(\rho_1 \cap \rho_2)$ is projective. Moreover, by [G, Proposition 1.3] ${}_BA$ is flat. However, since $Z(B) = Z(A) = F[x, y]$ the module ${}_BA$ is finitely generated and so ${}_BA$ is projective. Consequently, by [RS3, Theorem 5(ii)] we have

$$\begin{aligned}
\text{l.gl.dim } B &\leq \sup\left\{\text{l.gl.dim } A + \text{pr.dim}_B A,\ 1 + \text{l.gl.dim } \frac{B}{\rho}\right\} \\
&= \sup\{2 + 0, 1 + 0\} = 2,
\end{aligned}$$

where $\rho = \rho_1 \cap \rho_2$. Since $Z(B) = F[x, y]$ we must have l.gl.dim $B = 2$.

Let $C = M_2(B)$ and $I = M_2(\rho_1 \cap \rho_2)$. Hence ${}_CI$ is projective and $C/I \cong M_2(F \oplus F)$. Also, $I \cap Z(C) = I \cap F[x, y] = (\rho_1 \cap \rho_2) \cap F[x, y] = m_1 \cap m_2$. Consequently, $I = M_1 \cap M_2$ where M_i are maximal ideals in C, $M_i \cap Z(C) = m_i$ for $i = 1, 2$, and therefore $M_1 \neq M_2$.

For $i = 1, 2$ choose maximal left ideals λ_i of C such that $\lambda_i \supset M_i$. This is possible since for each i, $C/M_i \cong M_2(F)$. It is standard to show that $\lambda_i \cap Z(C) = M_i \cap Z(C) = m_i$ for $i = 1, 2$. Set $\lambda = \lambda_1 \cap \lambda_2$. Then $C/\lambda \cong C/\lambda_1 \oplus C/\lambda_2$. Therefore $\dim_F (C/\lambda) = 4$.

We claim that $\lambda C = C$. Otherwise there exists a maximal ideal T in C such that $T \supseteq \lambda C$. Then $T \supseteq \lambda_1 \cap \lambda_2 \supseteq M_1 \cap M_2$. So T contains M_1 or M_2—say $T = M_1$. However, $4 = \dim_F (C/M_1) \leq \dim_F (C/\lambda) = 4$ which implies that $\lambda = M_1 \subseteq \lambda_2$. This is a contradiction since $M_2 \subseteq \lambda_2$ and $M_1 + M_2 = C$.

We next claim that ${}_C\lambda$ is projective. Since $C/\lambda \cong C/\lambda_1 \oplus C/\lambda_2$ we have

$$\begin{aligned}\left(\frac{C}{\lambda}\right)^{(2)} &\cong \frac{C}{\lambda_1} \oplus \frac{C}{\lambda_1} \oplus \frac{C}{\lambda_2} \oplus \frac{C}{\lambda_2}\\ &\cong \frac{C}{M_1} \oplus \frac{C}{M_2}\\ &\cong \frac{C}{M_1 \cap M_2} = \frac{C}{I}.\end{aligned}$$

Now pr.dim$_C$ $(C/I) = 1$ since ${}_CI$ is projective. Consequently, pr.dim $(C/{}_C\lambda) = 1$ and ${}_C\lambda$ is projective.

Finally, define $R = F + \lambda$. It is clear that $Z(R) = F + \lambda \cap Z(Q(R)) = F + m_1 \cap m_2$. Evidently $Q(Z(R)) = Q(F[x, y])$. Now $F[x, y] = F + m_1 = F + m_2$ is integral over $F + m_1 \cap m_2 = Z(R)$. However, $Z(R) \neq F[x, y]$ which is the integral closure of $Z(R)$.

By the left-handed version of [RS3, Theorem 5(i)] we have gl.dim $R \leq$ sup{l.gl.dim C, $1 +$ l.gl.dim (R/λ) + pr.dim$_C$ $(C/\lambda\}$ = sup$\{2, 1 + 0 + 1\} = 2$. Now gl.dim $R > 1$ since otherwise by [MR, Theorem 13.9.16], $Z(R)$ would be a Dedekind domain and hence integrally closed.

Observe that $Z(R)$ is a Noetherian affine ring and hence R is finite over it.

Now let $m = m_1 \cap m_2$. Then m is a maximal ideal of $Z(R)$ and so height $(m) = 2$. However $F[x, y]m = m$. Hence there exists $q \in F[x, y] - Z(R)$ satisfying $qm \subseteq m$. Thus m has grade 1 and so $Z(R)$ is not a Cohen–Macaulay ring.

5. R Is Not Necessarily Integral Over $Z(R)$

Let F, K, L be fields where F and K are subfields of L such that $2 \leq [L : K] < \infty$, $2 \leq [L : F] < \infty$, and $\text{tr.}d_{K\cap F}\, L \geq 1$. (It is rather easy to find such fields, see e.g. [RSW].)

Let

$$A = \begin{pmatrix} L[x, y], & L[x, y] \\ (x, y), & L[x, y] \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} (x, y), & L[x, y] \\ (x, y), & L[x, y] \end{pmatrix}.$$

Consider

$$\rho = \begin{pmatrix} A, & A \\ N, & N \end{pmatrix}, \qquad S = \begin{pmatrix} A, & A \\ A, & A \end{pmatrix}.$$

Set

$$R = \begin{bmatrix} F + \rho, & \rho \\ \rho, & K + \rho \end{bmatrix}.$$

It is standard that gl.dim $A = 2 =$ gl.dim S, that N_A is projective, and, since $S/M_2(N) \cong (S/\rho)^{(2)}$, that ρ_S is projective. Hence $M_2(\rho)$ is a right projective $M_2(S)$-module satisfying $M_2(S)M_2(\rho) = M_2(S)$. Now by [RS3, Theorem 5(i)] we have

$$\text{r.gl.dim } R \leq \sup \left\{ \text{r.gl.dim } M_2(S),\ 1 + \text{r.gl.dim } \frac{R}{M_2(\rho)} + \text{pr.dim } \frac{M_2(S)}{M_2(\rho)} \right\}.$$

Now, since $R/M_2(\rho) \cong F \oplus K$, we conclude that gl.dim $R \leq 2$. To obtain $2 \leq$ gl.dim R we simply use the first inequality of [RS3, Theorem 5(i)].

Now, R is not integral over $Z(R)$, since $F \oplus K \cong R/M_2(\rho)$ is not algebraic over $F \cap K \cong Z(R)/(M_2(\rho) \cap Z(R))$. Finally, by standard arguments one shows R to be Noetherian by showing first that $Z(\left[\begin{smallmatrix} L+\rho, & \rho \\ \rho, & K+\rho \end{smallmatrix}\right])$ is affine (over K) and hence Noetherian, implying that $\left[\begin{smallmatrix} L+\rho, & \rho \\ \rho, & K+\rho \end{smallmatrix}\right]$ is Noetherian. Then the fact that

$$L \oplus K \cong \begin{bmatrix} L + \rho, & \rho \\ \rho, & K + \rho \end{bmatrix} \Big/ M_2(\rho) \supset \frac{R}{M_2(\rho)} \cong F \oplus K$$

is a finite (central) extension shows that R is Noetherian.

Consequently we have established the following.

EXAMPLE 15. There exists a P.I. ring with the following properties:

(i) R is prime and Noetherian and gl.dim $R = 2$.

(ii) R is not integral over $Z(R)$.

6. $Z(R)$ IS SOMETIMES INTEGRALLY CLOSED

The main concern of the present section is obtaining positive results with regard to Vasconcelos's question. In particular, if p.i.deg$(R) = 2$ and R is affine the answer to the question is positive.

We are indebted to J. T. Stafford for the following very short proof of the next lemma.

LEMMA 16. *Let R be a semiprime F.B.N. ring and M a maximal ideal in R. Suppose* gl.dim $R = 2$. *Then the following are equivalent:*

(i) $M^* \supset R$

(ii) *M_R is projective.*

Proof. As $\text{gl.dim}(R) = 2$, Bass's lemma [Ba] implies that M is projective if and only if M is reflexive if and only if $M^{*+} = M$. Since M is maximal, either $M = M^{*+}$ or $M^{*+} = R$, which implies that $M^* = R$.

We need the following (probably) well-known lemma, a proof of which can be found, for example, in [BV, Lemma 4.9].

LEMMA 17. *Let $A \subset B$ be P.I. rings. Suppose that B is prime and* $\text{p.i.deg}(B) = \text{p.i.deg}(A/P)$, *for some prime P in A. Then A is a prime ring and* $\text{p.i.deg}(A) = \text{p.i.deg}(B)$.

Our next result is the promised positive result with regard to Vasconcelos's question.

THEOREM 18. *Let R be a prime Noetherian affine P.I. ring. Suppose that*

(i) $\text{gl.dim } R = \text{K.dim } R = 2$.

(ii) $\text{p.i.deg}(R/M) \geq \frac{1}{2}\, \text{p.i.deg}(R)$, *for every maximal ideal M in R.*

Then $Z(R)$ is a Krull domain and R is a finite $Z(R)$-module.

Proof. That R is a finite $Z(R)$-module follows from Theorem A; however, the fact that $Z(R)$ is a Krull domain implies it anyway. The proof is by induction on $t(R)$, following the argument in Theorem A. The case $t(R) = 0$ is treated exactly the same. Likewise, we may assume the existence of a maximal ideal M in R, such that M_R is projective and $M^*M = M$. As before, we have $\text{gl.dim } M^* = 2$ and $t(M^*) < t(R)$. Moreover, M^*, being a left finitely generated R module, is an affine Noetherian ring. Also, by standard results $\text{K.dim } M^* = \text{K.dim } R = 2$, and (ii) is easily verified for M^*. Consequently, by induction, we may assume that $Z(M^*)$ is a Krull domain.

If $M^+ = R$ then $Z(R) = Z(M^+) = Z(M^*)$, where the last equality is an easy exercise. Consequently, $Z(R)$ is a Krull domain and the result follows. We may therefore assume that $M^+ \supset R$ and, by Lemma 16, that ${}_RM$ is projective. Now ${}_RM^*$ is finitely generated and by [CS] M^*_R is finitely generated. Consequently M^*/M is a finitely generated right R/M module. Hence M^*/M is an Artinian right R-module and, by a result of Lenagan (e.g., [J, Theorem 5.2.10]), M^*/M is an Artinian left M^*-module. Let $\lambda \subseteq M^*/M$ be a simple left M^*-submodule and $N \equiv l\text{-ann}_{M^*}\lambda$. Clearly, N is a two-sided maximal ideal in M^* and $M^*/N \cong \lambda^{(t)}$ for some t. Therefore $\text{pr.dim}_{M^*}(M^*/N) = \text{pr.dim}\ {}_{M^*}\lambda$. Also, $MM^* = M^*$ implies (by, e.g., [G, Proposition 2.1]) that $\text{pr.dim}_R M = \text{pr.dim}_{M^*} M$; that is, that ${}_{M^*}M$ is projective. Consequently $\text{pr.dim}_{M^*}(M^*/M) \leq 1$ and, if $\text{pr.dim}_{M^*}\lambda \geq 2$, the exact sequence $0 \to \lambda \to M^*/M \to (M^*/M)/\lambda \to 0$ leads to the contradiction $\text{pr.dim}_{M^*}\ (M^*/M)/\lambda) \geq 3$. Consequently, $\text{pr.dim}_{M^*}\lambda \leq 1$ and therefore pr.dim$_{M^*}(M^*/N) \le 1$. That is, $_{M^*}N$ is projective. Now M_R being projective implies that $_RM^*$ is projective. Hence, by transitivity, $_RN$ is projective.

We next claim that $_RN \cap R$ is projective. Suppose by negation that pr.dim$_R$ $N \cap R \geq 1$. Then the exact sequence

$$0 \to N \cap R \to N \oplus R \to N + R \to 0$$
$$(x, y) \to x + y.$$

leads to pr.dim$_R(N + R) \geq 2$. Since $_RM^*$ is left finitely generated, we find $\delta \in Z(R)$, $\delta \neq 0$, so that $M^*\delta \subseteq R$, and consequently $(N + R)\delta \subseteq R$. Now $_R(N + R) \cong (N + R)\delta$ and therefore pr.dim$_R(N + R)\delta \geq 2$. Consequently $(N + R)\delta \subset R$ and pr.dim$_R(R/(N + R)\delta) \geq 3$, a contradiction.

By [BS, Theorem 6.8], (ii) actually says that if P is a maximal ideal in R then either p.i.deg(R/P) = p.i.deg(R) or p.i.deg(R/P) = $\frac{1}{2}$ p.i.deg(R), and the same obviously holds for M^*.

We next claim that p.i.deg(M^*/N) = $\frac{1}{2}$ p.i.deg(M^*). Suppose by negation that p.i.deg (M^*/N) = p.i.deg(M^*). Then M^*_n is Azumaya with the unique maximal N_n, by the Artin–Procesi theorem, where $n \equiv N \cap Z(M^*)$. Now $_{M^*}N$ being projective implies that $_{M^*_n}N_n$ is projective and therefore gl.dim $M^*_n = 1$. Consequently, by [RS1], K.dim $M^*_n = 1$ and height(N_n) = 1. Hence height (N) = 1. Now, N being maximal, M^* being affine, and height (N) = 1 contradicts the catenary property of M^*, since K.dim $M^* = 2$.

Consequently, by Lemma 17, $P \equiv N \cap R$ is a prime ideal in R and p.i.deg ($R/(N \cap R)$) = $\frac{1}{2}$ p.i.deg(R). Moreover, since R is finite over $Z(R)$ by Theorem A, M^* is a finite $Z(R)$-module which implies that M^*/N is finite dimensional over $Z(R)/(N \cap Z(R))$. Therefore R/P is finite dimensional over $Z(R)/(N \cap Z(R))$; that is, P is a maximal ideal in R. We next claim that $P \neq M$. Indeed, if $P = M$, then $1 \in MM^* = PM^* \subseteq NM^* \subseteq N$, a contradiction.

Now $_RP$ is projective and height (P) = 2 since P is maximal and R is affine. Therefore $PP^+ = P$ (otherwise, by [H], height (P) = 1, a contradiction), and similarly $MM^+ = M$. So by induction, since $t(M^+)$, $t(P^+)$ are smaller than $t(R)$, we get that $Z(M^+)$, $Z(P^+)$ are Krull domains. Now since $P + M = R$ we get

$$M^+ \cap P^+ = R(M^+ \cap P^+) \subseteq (P + M)(M^+ \cap P^+) \subseteq P + M = R.$$

Hence $Z(R) = Z(P^+) \cap Z(M^+)$, which implies that $Z(R)$ is completely integrally closed. Finally, since $Z(R)$ is Noetherian (being affine), we conclude that $Z(R)$ is a Krull domain. Q.E.D.

COROLLARY 19. *Let R be a prime Noetherian affine P.I. ring. Suppose that* p.i.deg (R) = K.dim R = gl.dim R = 2. *Then $Z(R)$ is integrally closed.*

Our next example shows that the assumption gl.dim R = K.dim $R = 2$ is necessary in the previous theorem.

EXAMPLE 20. There exists a prime affine Noetherian P.I. ring, with the following properties:

(i) p.i.deg$(R) = 2 =$ gl.dim R.

(ii) K.dim $(R) = 1$.

(iii) $Z(R)$ is not integrally closed.

Proof. Let F be a field and k a proper subfield such that $[F : k] < \infty$. Consider $S = M_2(F[x])$, the 2×2 matrices over $F[x]$. Let $\rho = \left(\begin{smallmatrix} F[x] & F[x] \\ xF[x] & xF[x] \end{smallmatrix}\right)$. Obviously ρ is a maximal right ideal in S and $S\rho = S$.

Let

$$R \equiv k + \rho = \left\{ \begin{pmatrix} a + F[x], & F[x] \\ xF[x], & a + xF[x] \end{pmatrix} \middle| \, a \in k \right\}.$$

Then obviously $Z(R) = k + xF[x]$. Moreover, $F[x]$ is a finite $Z(R)$-module, which implies, by Eakin's theorem, that $Z(R)$ is Noetherian (actually affine over k) and K.dim $Z(R) = 1$. Moreover, $Q(F[x]) = Q(Z(R))$ implies that the integral closure of $Z(R)$ is $F[x]$. Hence, $Z(R)$ is not integrally closed.

That K.dim $R = 1$ follows now, since R is a finite $Z(R)$ module. Finally, by [RS3, Theorem 5(ii)] we have that

$$\text{l.gl.dim}\, R \leq \text{l.gl.dim}\, S + \text{l.gl.dim}\, R/\rho + 1 \leq 1 + 0 + 1 = 2.$$

Now, if l.gl.dim R = gl.dim $R = 1$ we get, by [RS1], that $Z(R)$ is a Dedekind domain, hence integrally closed, a contradiction. Consequently gl.dim $R = 2$.

Our next example shows that the affine assumption in theorem E, is necessary. Actually, this example serves as well as a counterexample to Vasconcelos's question.

EXAMPLE 21. There exists a prime Noetherian P.I. ring, with the following properties.

(i) gl.dim R = K.dim R = p.i.deg$(R) = 2$.

(ii) $Z(R)$ is Noetherian and so R is a finite $Z(R)$-module.

(iii) $Z(R)$ is not integrally closed.

Proof. This is really an elaboration of the previous example, so we omit the detail when it is similar. Let $A \subset B$ be commutative Noetherian domains satisfying the following properties:

(1) B is a finite A-module and $Q(B) = Q(A)$,

(2) gl.dim B = K.dim B = 2,

(3) B has a principal maximal ideal n such that $B/n \supset A/(A \cap n)$.

Let $\rho = \begin{bmatrix} B, & B \\ n, & n \end{bmatrix}$, $S = \begin{bmatrix} B, & B \\ B, & B \end{bmatrix}$ and define

$$R \equiv A + \rho = \left\{ \begin{bmatrix} x+B, & B \\ n, & x+n \end{bmatrix} \middle| \, x \in A \right\}.$$

Clearly ρ is a maximal generative right ideal in S so, by [RS3, Theorem 5(*i*)], gl.dim $R = 2$. Also $Z(R) = A + n$ Noetherian, $Q(Z(R)) = Q(B)$, and $Z(R)$ is not integrally closed by (3), as is easily verified.

Finally to exhibit such a pair A and B, let k be a proper subfield of a field F, with $[F : k] < \infty$. Set $T \equiv k[x, y] - \{(x-1)k[x, y] + yk[x, y] \cup xk[x, y]\}$. Then $B = F[x, y]_T$, $A = k[x, y]_T$, and $n = xF[x, y]_T$ provide such a pair.

Remark. It can be verified that $T(R) \neq R$ in the previous example. We omit the proof.

7. A Three-Dimensional Counterexample

We construct here the following

EXAMPLE 22. There exists a prime affine Noetherian P.I. ring with the following properties:

(1) K.dim R = 3 = gl.dim R.

(2) R is not a finite module over its center.

Proof. Let x, y, z be three commuting variables over the field k. Set $C = k[x, z]$, $I = (x, z)$. Let

$$\lambda = \begin{pmatrix} I, & C \\ I, & C \end{pmatrix}, \qquad A = \begin{pmatrix} C, & C \\ I, & C \end{pmatrix}, \qquad \rho = \begin{pmatrix} A, & A, & A \\ \lambda, & \lambda, & \lambda \\ \lambda, & \lambda, & \lambda \end{pmatrix}$$

and

$$a = \begin{pmatrix} y & & & & & \\ & y & & & & \\ & & y^2 & & & \\ & & & y & & \\ & & & & y & \\ & & & & & y \end{pmatrix}.$$

Then $R = k[a] + \rho[y]$ is the required example. Indeed, by standard arguments one observes that R is prime and affine. Now, to show that R is Noetherian, we need only show that $R[x, y, z]$, being a central extension of R, is integral over R. Clearly $x, z \in \rho[y] \subseteq R$, and $(y - a)(y^2 - a) = 0$ shows that y is integral over $k[a] \subseteq R$. Next, by inspecting the contributions to a central element in the (3, 3) and (5, 5) matrix positions, one observes that $Z(R) = k + (\rho[y] \cap Z(R)) = k + I[y] = k + xk[x, y, z] + zk[x, y, z]$. Now, since $k[x, y, z]$ is integral over R, the finiteness of R over $Z(R)$ would imply the integrality of $k[x, y, z]$ over $Z(R)$. An obvious contradiction.

Finally it is clear that k.dim $R = 3$, so by [GS] it suffices to show that gl.dim $R \leq 3$. This is proved by using [RS3] once more, taking into account that ρ is a projective right ideal in $M_3(A)$ and $M_3(A)\rho = M_3(A)$. Thus $\rho[y]$ is a right projective ideal in $M_3(A[y])$ and $M_3(A[y])\rho[y] = M_3(A[y])$. Now gl.dim $A = 2$ implies that gl.dim $M_3(A[y]) = 3$. Consequently by [RS3, Theorem 5], we have gl.dim $R \leq \sup\{\text{gl.dim } M_3(A[y]), 1 + \text{gl.dim } (R/\rho[y]) + \text{pr.dim } (M_3(A[y])/\rho[y])\}$. Hence gl.dim $R \leq \sup\{3, 1 + 1 + 1\} = 3$, as claimed.

References

[BGL] D. Baer, N. Geigle, and H. Lenzing, The preprojective algebra of a tame hereditary Artin algebra, *Comm. Algebra* **15** (1987), 425–457.

[Ba] H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, *Trans. Amer. Math. Soc.* **95** (1960), 466–488.

[BS] G. M. Bergman and L. W. Small, P.I. degrees and prime ideals, *J. Algebra* **33** (1975), 435–462.

[B1] A. Braun, A note on Noetherian PI rings, *Proc. Amer. Math. Soc.* **83** (1981), 670–672.

[B2] A. Braun, A characterization of prime Noetherian P.I. rings and a theorem of Mori–Nagata, *Proc. Amer. Math. Soc.* **74** (1979), 9–15.

[BrW] A. Braun and R. B. Warfield, Jr., Symmetry and localization in Noetherian prime P.I. rings, *J. Algebra* **118** (1988), 322–334.

[BV] A. Braun and N. Vonessen, Integrality for PI rings, *J. Algebra* **151** (1992), 39–79.

[BH] K. A. Brown and C. R. Hajarnavis, Homologically homogeneous rings, *Trans. Amer. Math. Soc.* **281** (1984), 197–208.

[BW] K. A. Brown and R. B. Warfield, Krull and global dimensions of fully bounded Noetherian rings, *Proc. Amer. Math. Soc.* **92** (1984), 169–174.

[CS] B. Cortzen and L. W. Small, Finite extensions of rings, *Proc. Amer. Math. Soc.* **103** (1988), 1058–1062.

[DR] V. Dlab and C. M. Ringel, The preprojective algebra of a modulated graph, *in* "Representation Theory II," Springer Lecture Notes in Mathematics, Vol. 832, pp. 216–231, Springer-Verlag, Berlin/New York, 1980.

[ER] D. Eisenbud and J. C. Robson, Hereditary Noetherian prime rings, *J. Algebra* **16** (1970), 86–104.

[G] K. R. Goodearl, Subrings of idealizer rings, *J. Algebra* **33** (1975), 405–429.

[GS] K. R. Goodearl and L. W. Small, Krull versus global dimension in Noetherian P.I. rings, *Proc. Amer. Math. Soc.* **92** (1984), 175–177.

[GW] K. R. GOODEARL AND R. B. WARFIELD, "An Introduction to Noncommutative Noetherian Rings," Cambridge Univ. Press, Cambridge, 1989.

[H] C. R. HAJARNAVIS, One-sided invertibility and localisation, *Glasgow Math J.* **34** (1992), 333–339.

[H1] C. R. HAJARNAVIS, One-sided invertibility and localisation, II, preprint.

[J] A. V. JATEGAONKAR, "Localisation in Noetherian Rings," London Math. Soc. Lecture Note Series, Vol. 98, Cambridge Univ. Press, Cambridge, 1986.

[K] I. KAPLANSKY, "Fields and Rings," Univ. of Chicago Press, Chicago, 1970.

[MR] J. C. MCCONNELL AND J. C. ROBSON, "Noncommutative Noetherian Rings," Wiley, New York, 1987.

[R] J. RAINWATER, Global dimension of fully bounded Noetherian rings, *Comm. Algebra* **15** (1987), 2143–2156.

[RSW] R. RESCO, L. W. SMALL, AND A. R. WADSWORTH, Tensor products of division rings and finite generation of subfields, *Proc. Amer. Math. Soc.* **77** (1979), 7–10.

[Ro] ROBSON, J. C. Some constructions of rings of finite global dimension, *Glasgow Math. J.* **26** (1985), 1–12.

[RS1] J. C. ROBSON AND L. W. SMALL, Hereditary prime PI rings are classical hereditary orders, *J. London Math. Soc.* **8** (1974), 499–503.

[RS2] J. C. ROBSON AND L. W. SMALL, Idempotent ideals in PI rings, *J. London Math. Soc.* **14** (1976), 120–122.

[RS3] J. C. ROBSON AND L. W. SMALL, Another change of rings theorem, *Bull. London Math. Soc.* **20** (1988), 297–301.

[Sch] W. SCHELTER, Integral extensions of rings satisfying a polynomial identity, *J. Algebra* **40** (1976), 245–257.

[St] J. T. STAFFORD, Global dimensions of semiprime Noetherian rings, *in* "Séminaire d'Algèbre" (M. P. Malliavin, Ed.), Springer Lecture Notes in Mathematics, Vol. 1296, pp. 247–260, Springer-Verlag, Berlin/New York, 1986.

[V] W. V. VASCONCELOS, "On Quasi Regular Algebras, "Sympos. Math. XI, pp. 11–22, Academic Press, London, 1973.